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On the stability of the quadratic mapping in Banach modules

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Abstract

We prove the generalized Hyers–Ulam–Rassias stability of the quadratic mapping in Banach modules over a unital Banach algebra.

Keywords: Stability; B-quadratic; Banach algebra

1. Introduction

In 1940, Ulam [12] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let E_1 and E_2 be Banach spaces. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geqslant 0$ and $p \in [0, 1)$ such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \leqslant \epsilon \left(\|x\|^p + \|y\|^p \right)$$

for all $x, y \in E_1$. Rassias [8] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

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$$\|f(x) - T(x)\| \leqslant \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E_1$.

A mapping $f : E_1 \to E_2$ is called quadratic if f satisfies the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in E_1$. Skof [11] was the first author to treat the Hyers–Ulam stability of a quadratic functional equation. Czerwik [4] generalized the Skof's result.

Let $f : E_1 \to E_2$ be a mapping with $f(0) = 0$ and satisfying the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \varphi(x, y)$$

for all $x, y \in E_1$. Assume that one of the series

$$\text{(i)}\quad \sum_{n=1}^{\infty} 2^{-2n}\varphi\bigl(2^{n-1}x, 2^{n-1}x\bigr) \quad \text{and}$$
$$\text{(ii)}\quad \sum_{n=1}^{\infty} 2^{2n-2}\varphi\bigl(2^{-n}x, 2^{-n}x\bigr)$$

converges for every $x \in E_1$ and denote by $\tilde{\varphi}(x)$ its sum. If, for every $x, y \in E_1$, as $n \to \infty$,

$$\text{(i)}\quad 2^{-2n}\varphi\bigl(2^{n-1}x, 2^{n-1}y\bigr) \to 0 \quad \text{or}$$
$$\text{(ii)}\quad 2^{2n-2}\varphi\bigl(2^{-n}x, 2^{-n}y\bigr) \to 0,$$

respectively, then there exists a unique quadratic mapping $Q : E_1 \to E_2$ such that

$$\|f(x) - Q(x)\| \leqslant \tilde{\varphi}(x)$$

for all $x \in E_1$. See [2] for details.

The stability problems of quadratic functional equations have been investigated in several papers [2–6,9].

Throughout this paper, let B be a unital Banach algebra with norm $|\cdot|$, $B_1 = \{a \in B \mid |a| = 1\}$, and let ${}_BM_1$ and ${}_BM_2$ be left Banach B-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively (see [1,10]).

In this paper, we are going to prove the generalized Hyers–Ulam–Rassias stability of the quadratic mapping in Banach modules over a unital Banach algebra.

2. Stability of the quadratic mapping in Banach modules

A quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is called *B-quadratic* if

$$Q(ax) = a^2 Q(x)$$

for all $a \in B$ and all $x \in {}_BM_1$.

Theorem 2.1. *Let $f : {}_BM_1 \to {}_BM_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0, \infty)$ such that*

$$\tilde{\varphi}(x) < \infty,$$
$$\left\| f(ax + ay) + f(ax - ay) - 2a^2 f(x) - 2a^2 f(y) \right\| \leqslant \varphi(x, y)$$

for all $a \in B_1$ and all $x, y \in {}_BM_1$, where $\tilde{\varphi}(x)$ is as defined in the introduction. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_BM_1$, then there exists a unique B-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ such that

$$\text{(iii)} \quad \left\| f(x) - Q(x) \right\| \leqslant \tilde{\varphi}(x)$$

for all $x \in {}_BM_1$.

Proof. By [2, Theorem 2], it follows from the second inequality of the statement for $a = 1$ that there exists a unique quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ satisfying the inequality (iii). The mapping $Q : {}_BM_1 \to {}_BM_2$ was given by

$$Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n}$$

for all $x \in {}_BM_1$ if φ satisfies (i), and

$$Q(x) = \lim_{n \to \infty} 4^{n-1} f\left(\frac{x}{2^n}\right)$$

for all $x \in {}_BM_1$ if φ satisfies (ii). The mapping $Q : {}_BM_1 \to {}_BM_2$ is similar to the additive mapping T given in the proof of [8, Theorem]. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_BM_1$, by the same reasoning as the proof of [8, Theorem], the quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is $\mathbb{R}$-quadratic.

Let us prove the theorem for the case that φ satisfies (i). By the assumption, for each $a \in B_1$,

$$\left\| f\big(2^n ax\big) - 4a^2 f\big(2^{n-1}x\big) \right\| \leqslant \varphi\big(2^{n-1}x, 2^{n-1}x\big)$$

for all $x \in {}_BM_1$. Using the fact that for each $a \in B$ and each $z \in {}_BM_2$ $\|az\| \leqslant K|a| \cdot \|z\|$ for some $K > 0$, one can show that

$$\begin{aligned}\left\| a^2 f\big(2^n x\big) - 4a^2 f\big(2^{n-1}x\big) \right\| &\leqslant K|a|^2 \cdot \left\| f\big(2^n x\big) - 4 f\big(2^{n-1}x\big) \right\| \\ &\leqslant K\varphi\big(2^{n-1}x, 2^{n-1}x\big)\end{aligned}$$

for all $a \in B_1$ and all $x \in {}_BM_1$. So

$$\begin{aligned}
&\|f(2^n ax) - a^2 f(2^n x)\| \\
&\quad \leqslant \|f(2^n ax) - 4a^2 f(2^{n-1}x)\| + \|4a^2 f(2^{n-1}x) - a^2 f(2^n x)\| \\
&\quad \leqslant \varphi(2^{n-1}x, 2^{n-1}x) + K\varphi(2^{n-1}x, 2^{n-1}x)
\end{aligned}$$

for all $a \in B_1$ and all $x \in {}_BM_1$. Thus $2^{-2n}\|f(2^n ax) - a^2 f(2^n x)\| \to 0$ as $n \to \infty$ for all $a \in B_1$ and all $x \in {}_BM_1$. Hence

$$Q(ax) = \lim_{n\to\infty} \frac{f(2^n ax)}{2^{2n}} = \lim_{n\to\infty} \frac{a^2 f(2^n x)}{2^{2n}} = a^2 Q(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

Similarly, for the case that φ satisfies (ii), one can obtain that

$$Q(ax) = \lim_{n\to\infty} \frac{f(2^{-n} ax)}{2^{2-2n}} = \lim_{n\to\infty} \frac{a^2 f(2^{-n} x)}{2^{2-2n}} = a^2 Q(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

Since Q is $\mathbb{R}$-quadratic and $Q(ax) = a^2 Q(x)$ for each $a \in B_1$,

$$Q(ax) = Q\left(|a| \cdot \frac{a}{|a|}x\right) = |a|^2 \cdot Q\left(\frac{a}{|a|}x\right) = |a|^2 \cdot \frac{a^2}{|a|^2} \cdot Q(x) = a^2 Q(x)$$

for all $a \in B (a \neq 0)$ and all $x \in {}_BM_1$. And $Q(0x) = 0^2 Q(x)$ for all $x \in {}_BM_1$. So the unique quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is B-quadratic, as desired. □

Corollary 2.1. *Let E_1 and E_2 be complex Banach spaces and $f : E_1 \to E_2$ a mapping with $f(0) = 0$ for which there exists a function $\varphi : E_1 \times E_1 \to [0, \infty)$ such that*

$$\begin{aligned}
&\tilde{\varphi}(x) < \infty, \\
&\|f(\lambda x + \lambda y) + f(\lambda x - \lambda y) - 2\lambda^2 f(x) - 2\lambda^2 f(y)\| \leqslant \varphi(x, y)
\end{aligned}$$

for all $\lambda \in \mathbb{C}_1$ and all $x, y \in E_1$, where $\tilde{\varphi}(x)$ is as defined in the introduction. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-quadratic mapping $Q : E_1 \to E_2$ such that

$$\|f(x) - Q(x)\| \leqslant \tilde{\varphi}(x)$$

for all $x \in E_1$.

Proof. Since $\mathbb{C}$ is a unital Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. By Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ satisfying the inequality given in the statement. □

Remark 2.1. If the second inequality in the statement of Theorem 2.1 is replaced by

$$\left\| f(ax+y) + f(ax-y) - 2a^2 f(x) - 2f(y) \right\| \leqslant \varphi(x,y),$$

then

$$\begin{aligned}
&\left\| f(ax+ay) + f(ax-ay) - 2a^2 f(x) - 2f(ay) \right\| \leqslant \varphi(x,ay),\\
&\left\| f(ay+ax) + f(ay-ax) - 2a^2 f(y) - 2f(ax) \right\| \leqslant \varphi(y,ax),\\
&\left\| f(ay+ax) + f(ay-ax) - 2f(ay) - 2f(ax) \right\| \leqslant \varphi(ay,ax).
\end{aligned}$$

So

$$\begin{aligned}
&\left\| f(ax+ay) + f(ax-ay) - 2a^2 f(x) - 2a^2 f(y) \right\|\\
&\quad \leqslant \varphi(x,ay) + \varphi(y,ax) + \varphi(ay,ax),
\end{aligned}$$

hence the result does also hold.

We prove the generalized Hyers–Ulam–Rassias stability of another quadratic mapping in Banach modules over a unital Banach algebra.

Theorem 2.2. *Let $f : {}_BM_1 \to {}_BM_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0,\infty)$ such that*

$$\begin{aligned}
&\tilde{\varphi}(x) < \infty,\\
&\left\| a^2 f(x+y) + a^2 f(x-y) - 2f(ax) - 2f(ay) \right\| \leqslant \varphi(x,y)
\end{aligned}$$

for all $a \in B_1$ and all $x, y \in {}_BM_1$, where $\tilde{\varphi}(x)$ is as defined in the introduction. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_BM_1$, then there exists a unique B-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ satisfying the inequality (iii).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{R}$-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ satisfying the inequality (iii).

Let us prove the theorem for the case that φ satisfies (i). By the assumption, for each $a \in B_1$,

$$\left\| a^2 f\left(2^n x\right) - 4f\left(2^{n-1} ax\right) \right\| \leqslant \varphi\left(2^{n-1}x, 2^{n-1}x\right)$$

for all $x \in {}_BM_1$. So

$$\begin{aligned}
&\left\| a^2 f\left(2^n x\right) - f\left(2^n ax\right) \right\|\\
&\quad \leqslant \left\| a^2 f\left(2^n x\right) - 4f\left(2^{n-1} ax\right) \right\| + \left\| 4f\left(2^{n-1} ax\right) - f\left(2^n ax\right) \right\|\\
&\quad \leqslant \varphi\left(2^{n-1}x, 2^{n-1}x\right) + \varphi\left(2^{n-1}ax, 2^{n-1}ax\right)
\end{aligned}$$

for all $a \in B_1$ and all $x \in {}_BM_1$. Thus $2^{-2n} \|a^2 f(2^n x) - f(2^n ax)\| \to 0$ as $n \to \infty$ for all $a \in B_1$ and all $x \in {}_BM_1$. Hence

$$a^2 Q(x) = \lim_{n\to\infty} \frac{a^2 f(2^n x)}{2^{2n}} = \lim_{n\to\infty} \frac{f(2^n ax)}{2^{2n}} = Q(ax)$$

for all $a \in B_1$ and all $x \in {}_B M_1$.

Similarly, for the case that φ satisfies (ii), one can obtain that

$$a^2 Q(x) = \lim_{n\to\infty} \frac{a^2 f(2^{-n} x)}{2^{2-2n}} = \lim_{n\to\infty} \frac{f(2^{-n} ax)}{2^{2-2n}} = Q(ax)$$

for all $a \in B_1$ and all $x \in {}_B M_1$.

The rest of the proof is similar to the proof of Theorem 2.1. So the unique quadratic mapping $Q : {}_B M_1 \to {}_B M_2$ is B-quadratic, as desired. □

Theorem 2.3. *Let $f : {}_B M_1 \to {}_B M_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_B M_1 \times {}_B M_1 \to [0, \infty)$ such that*

$$\tilde{\varphi}(x) < \infty,$$
$$\left\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \right\| \leqslant \varphi(x, y),$$
$$\left\| f(ax) - a^2 f(x) \right\| \leqslant \varphi(x, x)$$

for all $a \in B_1$ and all $x, y \in {}_B M_1$, where $\tilde{\varphi}(x)$ is as defined in the introduction. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B M_1$, then there exists a unique B-quadratic mapping $Q : {}_B M_1 \to {}_B M_2$ satisfying the inequality (iii).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{R}$-quadratic mapping $Q : {}_B M_1 \to {}_B M_2$ satisfying the inequality (iii).

Assume that φ satisfies (i). Since $2^{-2n} \| f(2^n ax) - a^2 f(2^n x) \| \to 0$ as $n \to \infty$ for all $a \in B_1$ and all $x \in {}_B M_1$,

$$Q(ax) = \lim_{n\to\infty} \frac{f(2^n ax)}{4^n} = \lim_{n\to\infty} a^2 \frac{f(2^n x)}{4^n} = a^2 Q(x)$$

for all $a \in B_1$ and all $x \in {}_B M_1$.

Similarly, for the case that φ satisfies (ii), one can obtain that

$$Q(ax) = \lim_{n\to\infty} \frac{f(2^{-n} ax)}{2^{2-2n}} = \lim_{n\to\infty} \frac{a^2 f(2^{-n} x)}{2^{2-2n}} = a^2 Q(x)$$

for all $a \in B_1$ and all $x \in {}_B M_1$.

The rest of the proof is the same as the proof of Theorem 2.1. So the unique $\mathbb{R}$-quadratic mapping $Q : {}_B M_1 \to {}_B M_2$ is B-quadratic, as desired. □

As corollaries of Theorems 2.2 and 2.3, one can obtain similar results to Corollary 2.1.

Remark 2.2. If the second inequality in the statement of Theorem 2.1 is replaced by

$$\left\| f(ax+y)+f(ax-y)-2a^2 f(x)-2f(y)\right\| \leqslant \varphi(x,y),$$

then

$$\left\| f(ax+x)+f(ax-x)-2a^2 f(x)-2f(x)\right\| \leqslant \varphi(x,x),$$
$$\left\| f(ax+x)+f(ax-x)-2f(ax)-2f(x)\right\| \leqslant \varphi(ax,x).$$

So

$$\left\|2f(ax)-2a^2 f(x)\right\| \leqslant \varphi(x,x)+\varphi(ax,x),$$

hence the result does also hold as a corollary of Theorem 2.3.

Remark 2.3. Let B be a complex unital Banach $*$-algebra. Let $b=aa^*$, a^*a, or $(aa^*+a^*a)/2$. If the second inequality in the statement of Theorem 2.1 is replaced by

$$\left\| f(ax+ay)+f(ax-ay)-2bf(x)-2bf(y)\right\| \leqslant \varphi(x,y),$$

then there exists a unique quadratic mapping $Q : {}_B M_1 \to {}_B M_2$ satisfying the inequality (iii) such that

$$Q(ax)=bQ(x)$$

for all $a\in B$ and all $x\in {}_B M_1$. The proof is similar to the proof of Theorem 2.1.

3. Stability of the pexiderized quadratic mapping in Banach modules

In this section, we prove the generalized Hyers–Ulam–Rassias stability of the pexiderized quadratic mapping in Banach modules over a unital Banach algebra.

Theorem 3.1. *Let $\varphi : {}_B M_1 \times {}_B M_1 \to [0,\infty)$ be a function satisfying*

$$\text{(iv)}\quad \tilde{\varphi}(x,y) := \sum_{k=0}^{\infty} \frac{1}{4^{k+1}} \varphi\left(2^k x, 2^k y\right) < \infty, \quad \forall x,y\in {}_B M_1, \quad \textit{or}$$

$$\text{(v)}\quad \tilde{\varphi}(x,y) := \sum_{k=0}^{\infty} 4^k \varphi\left(\frac{x}{2^{k+1}}, \frac{y}{2^{k+1}}\right) < \infty, \quad \forall x,y\in {}_B M_1.$$

Suppose that the mappings $f,g : {}_B M_1 \to {}_B M_2$ satisfy the inequality

$$\left\| f(ax+ay)+f(ax-ay)-2a^2 g(x)-2a^2 g(y)\right\| \leqslant \varphi(x,y)$$

for all $a\in B_1$ and all $x,y\in {}_B M_1$. Assume that $f(0)=0$ for the case (v). *If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in {}_B M_1$, then there exists a unique B-quadratic mapping $Q : {}_B M_1 \to {}_B M_2$ such that*

$$\|f(x)-f(0)-Q(x)\| \leqslant \tilde{\varphi}(x,x)+2\tilde{\varphi}(x,0)+\tilde{\varphi}(0,0),$$
$$\|g(x)-g(0)-Q(x)\| \leqslant \frac{1}{2}\tilde{\varphi}(2x,0)+\tilde{\varphi}(x,x)+\frac{1}{2}\tilde{\varphi}(0,0)$$

for all $x \in {}_BM_1$.

Proof. For $a=1$, it was shown in [7, Theorem 2.2] that there exists a unique quadratic mapping $Q: {}_BM_1 \to {}_BM_2$ satisfying the inequalities given in the statement. The mapping $Q: {}_BM_1 \to {}_BM_2$ was given by

$$Q(x)=\lim_{n\to\infty}\frac{f(2^nx)}{4^n}=\lim_{n\to\infty}\frac{g(2^nx)}{4^n}$$

for all $x \in {}_BM_1$ if φ satisfies (iv), and

$$Q(x)=\lim_{n\to\infty}4^nf\left(\frac{x}{2^n}\right)=\lim_{n\to\infty}4^ng\left(\frac{x}{2^n}\right)$$

for all $x \in {}_BM_1$ if φ satisfies (v). By the same reasoning as the proof of Theorem 2.1, the quadratic mapping $Q: {}_BM_1 \to {}_BM_2$ is $\mathbb{R}$-quadratic.

Next, assume that φ satisfies (iv). By the assumption, for each $a \in B_1$,

$$\|f(2^nax)-4a^2g(2^{n-1}x)\| \leqslant \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in {}_BM_1$. Using the fact that for each $a \in B$ and each $z \in {}_BM_2$ $\|az\| \leqslant K|a|\cdot\|z\|$ for some $K>0$, one can show that

$$\begin{aligned}\|a^2f(2^nx)-4a^2g(2^{n-1}x)\| &\leqslant K|a|^2\cdot\|f(2^nx)-4g(2^{n-1}x)\|\\ &\leqslant K\varphi(2^{n-1}x, 2^{n-1}x)\end{aligned}$$

for all $a \in B_1$ and all $x \in {}_BM_1$. So

$$\begin{aligned}&\|f(2^nax)-a^2g(2^nx)\|\\ &\quad\leqslant \|f(2^nax)-4a^2g(2^{n-1}x)\|+\|4a^2g(2^{n-1}x)-a^2f(2^nx)\|\\ &\quad\leqslant \varphi(2^{n-1}x, 2^{n-1}x)+K\varphi(2^{n-1}x, 2^{n-1}x)\end{aligned}$$

for all $a \in B_1$ and all $x \in {}_BM_1$. Thus $2^{-2n}\|f(2^nax)-a^2g(2^nx)\| \to 0$ as $n\to\infty$ for all $a \in B_1$ and all $x \in {}_BM_1$. Hence

$$Q(ax)=\lim_{n\to\infty}\frac{f(2^nax)}{2^{2n}}=\lim_{n\to\infty}\frac{a^2g(2^nx)}{2^{2n}}=a^2Q(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

Similarly, for the case that φ satisfies (v), one can obtain that

$$Q(ax)=\lim_{n\to\infty}\frac{f(2^{-n}ax)}{4^{-n}}=\lim_{n\to\infty}\frac{a^2g(2^{-n}x)}{4^{-n}}=a^2Q(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

The rest of the proof is the same as the proof of Theorem 2.1. So the unique quadratic mapping $Q: {}_BM_1 \to {}_BM_2$ is B-quadratic. □

Remark 3.1. (1) When the inequality

$$\left\| f(ax+ay) + f(ax-ay) - 2a^2 g(x) - 2a^2 g(y) \right\| \leqslant \varphi(x, y)$$

in the statement of Theorem 3.1 is replaced by

$$\left\| a^2 f(x+y) + a^2 f(x-y) - 2g(ax) - 2g(ay) \right\| \leqslant \varphi(x, y),$$

the result does also hold. The proof is similar to the proof of Theorem 2.2.

(2) Let $\eta(x)$ and $\zeta(x)$ be $f(x)$ or $g(x)$. When the inequality

$$\left\| f(ax+ay) + f(ax-ay) - 2a^2 g(x) - 2a^2 g(y) \right\| \leqslant \varphi(x, y)$$

in the statement of Theorem 3.1 is replaced by

$$\left\| f(x+y) + f(x-y) - 2g(x) - 2g(y) \right\| \leqslant \varphi(x, y),$$
$$\left\| \eta(ax) - a^2 \zeta(x) \right\| \leqslant \varphi(x, x),$$

the result does also hold. The proof is similar to the proof of Theorem 2.3.

Similarly, one can prove the stability of the other quadratic mappings in Banach modules over a unital Banach algebra.

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